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# A NOTE ON FAXEN LAWS FOR NONSPHERICAL PARTICLES

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**Abstract**—The advantages of new forms of Faxen laws with derivatives of finite order are discussed and specifically illustrated for prolate spheroids. Earlier ideas on the connection between the Faxen laws and certain "associated solutions" (single-particle solutions of the Stokes flow where the ambient velocity is a uniform stream or linear field) have been refined for the case where the associated solution has a relatively simple singularity solution.

## 1. INTRODUCTION

In this note, we derive new forms for the Faxen law for the force, torque and stresslet on a particle of arbitrary shape and give specific examples for the prolate spheroid. The companion paper utilizes advantages of the new forms. Faxen (1924) showed that the force, $\mathbf{F}$ and torque, $\mathbf{T}$ on a stationary, rigid sphere in an unbounded fluid in creeping flow $\mathbf{v}^\infty$ are

$$\mathbf{F} = 6\pi\mu a(1 + \tfrac{1}{6} a^2\nabla^2)\mathbf{v}^\infty(\mathbf{x}_c), \qquad [1.1]$$

$$\mathbf{T} = 4\pi\mu a^3\nabla \times \mathbf{v}^\infty(\mathbf{x}_c). \qquad [1.2]$$

Batchelor & Green (1972) have derived an analogous Faxen-type relation for the stresslet or symmetric stress dipole $\mathbf{S}$ for a rigid sphere in an ambient rate-of-strain field $\mathbf{e}^\infty$,

$$\mathbf{S} = \tfrac{20}{3}\pi\mu a^3(1 + \tfrac{1}{10} a^2\nabla^2)\, \mathbf{e}^\infty(\mathbf{x}_c). \qquad [1.3]$$

As shown by Felderhof (1977) and Rallison (1977) these relations can be used in the multipole expansion of the method-of-reflections calculations for hydrodynamic interactions between two or more particles.

Generalization of the Faxen laws to other shapes, e.g. ellipsoids have been carried out by Brenner (1964) and Rallison (1978) and are of the form

$$\mathbf{F} \propto \left(1 + \frac{1}{3!}D^2 + \frac{1}{5!}D^4 + \cdots\right)\mathbf{v}^\infty(\mathbf{x}_c), \qquad [1.4]$$

$$\mathbf{T} \propto \left(1 + \frac{2.3!}{5!}D^2 + \frac{3.3!}{7!}D^4 + \cdots\right)\nabla \times \mathbf{v}^\infty(\mathbf{x}_c), \qquad [1.5]$$

$$\mathbf{S} \propto \left(1 + \frac{2.3!}{5!}D^2 + \frac{3.3!}{7!}D^4 + \cdots\right)\mathbf{e}^\infty(\mathbf{x}_c), \qquad [1.6]$$

where $D^2 = a^2\partial^2/\delta x^2 + b^2\partial^2/\delta y^2 + c^2\partial^2/\partial z^2$.
Note that the loss of isotropy introduces derivatives of higher order than that present for spheres [see also Brenner & Haber (1983)].

We would like to eliminate the infinite series so that:

(1) The method-of-reflections can be applied to ambient flow fields obtained via numerical solutions such as finite elements and finite differences.

(2) The reflection process can be simplified by keeping terms from each multipole–multipole interaction together (as shown in the following paper).

From Brenner (1964) and Rallison (1978), we know that, because of the reciprocal theorem, the Faxen law can be derived from information contained in certain associated single-particle solutions: the (Faxen) force law from the uniform stream solution; the torque and stresslet laws from the linear field solution. A more explicit statement can be made if the associated solution is represented in terms of the fundamental solution of the Stokes equation (i.e. a singularity solution). Hinch (1977) has noted (without proof) that the functional forms of the Faxen laws for a sphere are identical to the functional forms of the associated singularity solutions.

The appendix contains two proofs of the following proposition:

Suppose that the singularity solution for a stationary particle in a uniform stream, $\mathbf{U}^\infty$ is of the form:

$$\mathbf{v}(\mathbf{x}) = \mathbf{U}^\infty + \mathbf{U}^\infty \cdot \hat{L}_\xi \left\{ \mathbf{I} \frac{(\mathbf{x} - \xi)}{8\pi\mu} \right\},$$

where $\mathbf{I}$ is the Oseen tensor given by

$$\mathbf{I}(\mathbf{x}) = \frac{1}{|\mathbf{x}|}\delta + \frac{1}{|\mathbf{x}|^3}\mathbf{x}\mathbf{x},$$

and $\hat{L}$ is a linear functional. The parameter $\xi$ denotes the distribution of the singularities. Then the Faxen law for the force on a stationary particle in an ambient field $\mathbf{v}^\infty$ is

$$F = -\hat{L}_\xi \{\mathbf{v}^\infty(\xi)\}.$$

The result can be extended to include translational and rotational motions by adding the (known) contributions from these motions. Similar results exist relating the torque and stresslet laws with their associated singularity solutions.

These ideas will now be illustrated for prolate spheroids. The solutions of Chwang & Wu (1974 and 1975) and Youngren & Acrivos (1975) indicate that Faxen relations with derivatives of finite order can be constructed if one relaxes the restriction of evaluating $\mathbf{v}^\infty$ at only one position.

## 2. FAXEN LAW FOR PROLATE SPHEROIDS

The Chwang–Wu solutions for an isolated prolate spheroid in a uniform stream $\mathbf{U}^\infty$, vorticity field $\Omega$ and rate-of-strain field $\mathbf{E}$ can be rewritten as

$$\mathbf{v}(\mathbf{x}) = \mathbf{U}^\infty - \mathbf{U}^\infty \cdot (\alpha_1 \mathbf{dd} + \alpha_2(\delta - \mathbf{dd})) \cdot \int_{-c}^{c} \left(1 + (c^2 - \xi^2)\frac{(1-e^2)}{4e^2}\nabla^2\right) \mathbf{I}(\mathbf{x} - \xi) \mathrm{d}\xi, \qquad [2.1]$$

$$v_i(\mathbf{x}) = \epsilon_{ijk}\Omega_j x_k - \frac{1}{2}\Omega_l \epsilon_{jkm}\{\gamma d_l d_m + \gamma'(\delta_{lm} - d_l d_m)\} \int_{-c}^{c} (c^2 - \xi^2)\, I_{ij,k}(\mathbf{x} - \xi)\mathrm{d}\xi$$

$$+ \Omega_l d_m \epsilon_{klm} d_j \alpha' \int_{-c}^{c} (c^2 - \xi^2)\left\{1 + (c^2 - \xi^2)\frac{(1-e^2)}{8e^2}\nabla^2\right\} \qquad [2.2]$$

$$\cdot \frac{1}{2}\{I_{ij,k}(\mathbf{x} - \xi) + I_{ik,j}(\mathbf{x} - \xi)\}\mathrm{d}\xi \quad \text{with} \quad \Omega_i = \frac{1}{2}\epsilon_{ijk}\Omega_{jk}$$

and

$$
\begin{aligned}
v_i(\mathbf{x}) = E_{ij}x_j - E_{lm}\Bigg\{&\alpha_5 \frac{1}{2}\left(\left(d_jd_k - \underline{\frac{1}{3}\delta_{jk}}\right)\left(d_ld_m - \underline{\frac{1}{3}\delta_{lm}}\right)\right. \\
&+ \frac{1}{4}\alpha^* \left(d_j\delta_{km}d_l + d_j\delta_{kl}d_m + \delta_{jm}d_kd_l + \delta_{jl}d_kd_m - 4d_jd_kd_ld_m\right) \\
&+ \frac{1}{2}\alpha_4\left(\delta_{jl}\delta_{km} + \delta_{jm}\delta_{kl} - \underline{\delta_{jk}\delta_{lm}} + \underline{d_jd_k\delta_{lm}} + \underline{\delta_{jk}d_ld_m}\right. \\
&\left.\left.- d_j\delta_{km}d_l - \delta_{jm}d_kd_l - d_j\delta_{kl}d_m - \delta_{jl}d_kd_m + d_jd_kd_ld_m\right)\right)\Bigg\} \\
&\times \int_{-c}^{c}(c^2 - \xi^2)\left\{1 + (c^2 - \xi^2)\frac{(1 - e^2)}{8e^2}\nabla^2\right\} I_{ij,k}(\mathbf{x} - \boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} \\
&+ E_{lm}d_j\delta_{km}d_l\gamma^* \int_{-c}^{c}(c^2 - \xi^2)\frac{1}{2}\{I_{ij,k}(\mathbf{x} - \boldsymbol{\xi}) - I_{ik,j}(\mathbf{x} - \boldsymbol{\xi})\}\mathrm{d}\boldsymbol{\xi}.
\end{aligned}
\qquad [2.3]
$$

where again, $\mathbf{I}$ is the Oseen tensor. $a$ is the length of the major semi-axis, $c$ is the distance from the centroid to the focal point, $e = c/a$ is the eccentricity and $d$ is the orientation of the spheroidal axis. The shape factors $\alpha$ and $\gamma$ from Chwang & Wu are reproduced in table 1.

Consequently, the Faxen laws for the force, torque and stresslet on a particle moving as $\mathbf{U} + \omega \times \mathbf{x}$ can be written as

$$
\begin{aligned}
\mathbf{F} = 8\pi\mu\{\alpha_1\mathbf{dd} + \alpha_2(\boldsymbol{\delta} - \mathbf{dd})\}\cdot \int_{-c}^{c} \\
\cdot\left\{1 + (c^2 - \xi^2)\frac{(1 - e^2)}{4e^2}\nabla^2\right\}\mathbf{v}^\infty(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} - 16\pi\mu ae\{\alpha_1\mathbf{dd} + \alpha_2(\boldsymbol{\delta} - \mathbf{dd})\}\cdot\mathbf{U},
\end{aligned}
\qquad [2.4]
$$

$$
\begin{aligned}
\mathbf{T} = 4\pi\mu\{\gamma\mathbf{dd} + \gamma'(\boldsymbol{\delta} - \mathbf{dd})\}\cdot\int_{-c}^{c}(c^2 - \xi^2)\nabla\times\mathbf{v}^\infty(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} + 8\pi\mu\alpha'\mathbf{d}\times\int_{-c}^{c}(c^2 - \xi^2) \\
\cdot\left\{1 + (c^2 - \xi^2)\frac{(1 - e^2)}{8e^2}\nabla^2\right\}\mathbf{d}\cdot\mathbf{e}^\infty(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} - \frac{32}{3}\pi\mu a^3e^3\{\gamma\mathbf{dd} + \gamma'(\boldsymbol{\delta} - \mathbf{dd})\}\cdot\omega,
\end{aligned}
\qquad [2.5]
$$

and

$$
\begin{aligned}
S_{ij} = 8\pi\mu\Bigg\{&-\frac{1}{2}\alpha_5\left(d_id_j - \frac{1}{3}\delta_{ij}\right)\left(d_kd_l - \underline{\frac{1}{3}\delta_{kl}}\right) \\
&- \frac{1}{4}\alpha^*\left(d_i\delta_{jk}d_l + d_i\delta_{jl}d_k + \delta_{il}d_jd_k + \delta_{ik}d_jd_l - 4d_id_jd_kd_l\right. \\
&- \frac{1}{2}\alpha_4\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \underline{\delta_{ij}\delta_{kl}} + \underline{d_jd_j\delta_{kl}} + \delta_{ij}d_kd_l + d_id_jd_kd_l\right. \\
&\left.- d_i\delta_{jl}d_k - d_i\delta_{jk}d_l - \delta_{ik}d_jd_l - \delta_{il}d_jd_k\right)\Bigg\}\int_{-c}^{c}(c^2 - \xi^2) \\
&\cdot\left\{1 + (c^2 - \xi^2)\frac{(1 - e^2)}{8e^2}\nabla^2\right\}e_{kl}^\infty(\xi)\mathrm{d}\xi \\
&-2\pi\mu\gamma^*(d_i\epsilon_{jkl}d_l + d_j\epsilon_{ikl}d_l)\int_{-c}^{c}(c^2 - \xi^2)\{\nabla\times\mathbf{v}^\infty(\boldsymbol{\xi}) - 2\omega\}_k d\xi.
\end{aligned}
\qquad [2.6]
$$

Table 1. Constants for the velocity representation for the spheroid. (i) Constants derived from Chwang & Wu (1974, 1975)

$$\alpha_1 = e^2\left\{-2e + (1 + e^2)\log\left(\frac{1+e}{1-e}\right)\right\}^{-1}$$

$$\alpha_2 = 2e^2\left\{2e + (3e^2 - 1)\log\left(\frac{1+e}{1-e}\right)\right\}^{-1}$$

$$\gamma = (1 - e^2)\left\{2e - (1 - e^2)\log\left(\frac{1+e}{1-e}\right)\right\}^{-1}$$

$$\gamma_3 = (1 - e^2)\left\{-2e + (1 + e^2)\log\left(\frac{1+e}{1-e}\right)\right\}^{-1}$$

$$\gamma_3' = \gamma_3(1 - e^2)^{-1}$$

$$\alpha_3 = 2e^2\gamma_3\left\{-2e + \log\left(\frac{1+e}{1-e}\right)\right\}\left\{2e(2e^2 - 3) + 3(1 - e^2)\log\left(\frac{1+e}{1-e}\right)\right\}^{-1}$$

$$\alpha_3' = e^2\gamma_3'\left\{-2e + (1 - e^2)\log\left(\frac{1+e}{1-e}\right)\right\}\left\{2e(2e^2 - 3) + 3(1 - e^2)\log\left(\frac{1+e}{1-e}\right)\right\}^{-1}$$

$$\alpha_4 = 2e^2(1 - e^2)\left\{2e(3 - 5e^2) - 3(1 - e^2)^2\log\left(\frac{1+e}{1-e}\right)\right\}^{-1}$$

$$\alpha_5 = e^2\left\{6e - (3 - e^2)\log\left(\frac{1+e}{1-e}\right)\right\}^{-1}$$

$$\alpha' \equiv \alpha_3 + \alpha_3' = e^2\left\{-2e + (1 + e^2)\log\left(\frac{1+e}{1-e}\right)\right\}^{-1}$$

$$\gamma' \equiv \gamma_3 + \gamma_3' = (2 - e^2)\left\{-2e + (1 + e^2)\log\left(\frac{1+e}{1-e}\right)\right\}^{-1}$$

$$\alpha^* \equiv \alpha_3 - \alpha_3'$$

$$\gamma^* \equiv \gamma_3 - \gamma_3' = -e^2\left\{-2e + (1 + e^2)\log\left(\frac{1+e}{1-e}\right)\right\}^{-1}$$

These expressions may be reduced to earlier forms by expanding the ambient field in a Taylor series at the particle centroid. However, in the present form, the infinite series is replaced by integrals which are actually better for discrete data.

The translational and rotational motion of a force-free and torque-free spheroid follows from [2.4] to [2.6] as

$$\mathbf{U} = \frac{1}{2c}\int_{-c}^{c}\left\{1 + (c^2 - \xi^2)\frac{(1 - e^2)}{4e^2}\nabla^2\right\}\mathbf{v}^\infty(\xi)\,d\xi, \qquad [2.7]$$

$$\begin{aligned}\omega = \frac{3}{8c^3}\int_{-c}^{c}(c^2 - \xi^2)\,\nabla \times \mathbf{v}^\infty(\xi)\,d\xi \\ + \frac{3}{4c^3}\frac{e^2}{(2 - e^2)}\int_{-c}^{c}(c^2 - \xi^2)\left\{1 + (c^2 - \xi^2)\frac{(1 - e^2)}{8e^2}\nabla^2\right\}\mathbf{d} \times \mathbf{e}^\infty(\xi)\cdot\mathbf{d}\,d\xi.\end{aligned} \qquad [2.8]$$

One application of the last equation is the generalization of the Jeffery equations for the evolution of the orientation of a spheroid in a linear field [see Leal & Hinch (1972)] to the following for an arbitrary Stokes field,

$$\mathbf{d} - \frac{3}{4c^3}\int_{-c}^{c}(c^2 - \xi^2)\mathbf{\Omega}^{\infty}(\xi) \times \mathbf{d}\, d\xi + \frac{3}{4c^3}\frac{e^2}{(2 - e^2)}\int_{-c}^{c}(c^2 - \xi^2) \cdot \left\{1 + (c^2 - \xi^2)\frac{(1 - e^2)}{8e^2}\nabla^2\right\}(\mathbf{e}^{\infty}(\xi)\cdot\mathbf{d} - \mathbf{e}^{\infty}(\xi):\mathbf{ddd})d\xi. \qquad [2.9]$$

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## NOTATION

- $a$ length of major semiaxis of spheroid.
- $c$ distance from center to foci.
- $\mathbf{d}$ unit vector denoting orientation of spheroid axis.
- $e$ eccentricity of the spheroid.
- $\mathbf{e}, \mathbf{E}$ rate-of-strain tensor.
- $\mathbf{F}$ force exerted on the particle by the fluid.
- $\mathbf{I}$ Oseen tensor
- $\mathbf{S}$ stresslet or symmetric part of the stress dipole.
- $\mathbf{T}$ torque exerted on the particle by the fluid.
- $\mathbf{U}$ particle translational velocity.
- $\mathbf{v}$ velocity.
- $\mathbf{x}$ position vector.
- $\mathbf{x}_c$ position of particle centroid.

*Greek letters*

- $\alpha$ constants in the Chwang-Wu singularity solutions.
- $\gamma$ constants in the Chwang-Wu singularity solutions.
- $\boldsymbol{\delta}$ identity tensor.
- $\boldsymbol{\epsilon}$ alternating tensor.
- $\mu$ viscosity.
- $\boldsymbol{\xi}$ vector denoting position on the spheroid axis.
- $\boldsymbol{\omega}$ particle angular velocity.
- $\mathbf{\Omega}$ vorticity.
- $\Omega$ vorticity tensor.

*Subscripts*

*i,j,k,ℓ,m*

indices used in the Einstein summation convention.

*Superscripts*

- $\infty$ ambient field.

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## APPENDIX

In this appendix we prove that if the uniform streaming problem has the functional form:

$$\mathbf{v}(\mathbf{x}) = \mathbf{U}^{\infty} + \mathbf{U}^{\infty} \cdot \hat{L}_{\xi}\left\{\mathbf{I}\frac{(\mathbf{x}-\xi)}{8\pi\mu}\right\}$$

then the Faxen law for the force on the stationary particle is

$$\mathbf{F} = -\hat{L}_{\xi}\{\mathbf{v}^{\infty}(\xi)\}.$$

*Proof I*

We apply the reciprocal theorem of Lorentz (Happel & Brenner 1965) to two velocity-stress pairs $(\mathbf{v}^D, \sigma)$ and $(\mathbf{v}', \sigma')$ where the first set is the disturbance solution to the problem on interest and the second set is the singularity (disturbance) solution of the uniform streaming problem. The singularities are assumed to be outside the fluid region (i.e. inside or on the surface of the particle). The reciprocal theorem states that

$$\oint_{S_p} v_i^D \sigma'_{ij} n_j \, \mathrm{d}A = \oint_{S_p} v'_i \sigma^D_{ij} n_j \, \mathrm{d}A. \qquad \text{[A1]}$$

On the particle surface, $v'_i = -U_i^{\infty}$ so the RHS of [A1] simplifies to $-\mathbf{U}^{\infty} \cdot \mathbf{F}$.

We apply the divergence theorem to the analytical continuation of $\mathbf{v}^D$ inside the particle. The LHS of [A1] becomes

$$\int_{V_p} \frac{v_i^D}{x_j} \sigma'_{ij} \, \mathrm{d}V + \int_{V_p} v_i^{\infty} \hat{L}_{\xi}\{\delta(\mathbf{x}-\xi)\} \, \mathrm{d}V,$$

where we have used the Stokes equation to simplify $\nabla \cdot \sigma'$. The first term vanishes for a rigid particle. (Even in the case where the particle is rotating, this term still vanishes because the rate-of-strain vanishes inside so that $\nabla\mathbf{v}^D$ is antisymmetric.) The properties of the Dirac delta function simplifies the second term to: $\hat{L}_{\xi}\{(\mathbf{v}^{\infty}(\mathbf{x}-\xi)\}$. The Faxen law follows as

$$\mathbf{F} = -\hat{L}_{\xi}\{\mathbf{v}^{\infty}(\xi)\}.$$

*Proof II*

We start with the integral representation for the velocity of interest:

$$\mathbf{v}(\mathbf{x}) = \mathbf{v}^{\infty}(\mathbf{x}) - \frac{1}{8\pi\mu}\oint_{S_p} (\sigma \cdot n) \cdot \mathbf{I}(\mathbf{x} - \mathbf{x}_s)\, dS(\mathbf{x}_s), \quad \text{if } \mathbf{x} \text{ is in the fluid,}$$

$$\mathbf{v}^{\infty}(\mathbf{x}) = \frac{1}{8\pi\mu}\oint_{S_p} (\sigma \cdot n) \cdot \mathbf{I}(\mathbf{x} - \mathbf{x}_s)\, dS(\mathbf{x}_s), \quad \text{if } \mathbf{x} \text{ is in the particle.} \qquad [A2]$$

This can be derived by applying the reciprocal theorem to $(\mathbf{v},\sigma)$ and $(\mathbf{I},\Sigma)$, where $\Sigma$ is the stress field of $\mathbf{I}$. We also have, from the boundary condition of the uniform streaming problem,

$$\hat{L}_\xi\left\{\mathbf{I}\left(\frac{\mathbf{x} - \xi}{8\pi\mu}\right)\right\} = -\delta, \quad \text{if } \mathbf{x} \text{ is in the particle.}$$

The Faxen law is obtained by applying $\hat{L}_\xi$ to both sides of the second equation in [A2]. The Oseen tensor in the surface integral is converted into (the negative of) the idemfactor so that the integral reduces to the expression for the force.